

F-theorem, duality and SUSY breaking in one-adjoint Chern-Simons-Matter theories

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Abstract

We extend previous work on $\mathcal{N} = 2$ Chern-Simons theories coupled to a single adjoint chiral superfield using localization techniques and the F -maximization principle. We provide tests of a series of proposed 3D Seiberg dualities and a new class of tests of the conjectured F-theorem. In addition, a proposal is made for a modification of the F -maximization principle that takes into account the effects of decoupling fields. Finally, we formulate and provide evidence for a new general non-perturbative constraint on spontaneous supersymmetry breaking in three dimensions based on Q -deformed S^3 partition functions. An explicit illustration based on the known analytic solution of the Chern-Simons matrix model is presented.

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1. Introduction

Important progress has been recently achieved in understanding some of the key properties of three-dimensional supersymmetric conformal field theories (SCFTs). The main technical tool has been localization which (after proper regularization) allows to express the free energy of the Euclidean CFT on the three-sphere

$$F = -\log |Z_{S^3}| \tag{1.1}$$

in terms of a compact matrix integral expression [1–3].

The free energy F provides a variety of useful information. For example, three-dimensional SCFTs with $\mathcal{N} = 2$ supersymmetry have a conserved $U(1)$ R-symmetry that sits in the same

supermultiplet as the stress-energy tensor. In a general interacting SCFT this symmetry receives quantum corrections and the quantum numbers associated with it become non-trivial functions of the parameters that define the theory. The proper coupling of the matter fields to the curvature of S^3 makes F a function of the R-charges. It has been proposed that the exact $U(1)$ R-symmetry of the theory is that R-symmetry which locally maximizes F [2]. Non-trivial tests of this proposal have appeared in [4–10].

Secondly, there are by now many known examples of three-dimensional SCFTs that exhibit dualities analogous to the S-duality and Seiberg duality in four dimensions. Matching the free energy F of dual theories provides non-trivial checks of these dualities [11–15] (analogous checks of 3D dualities based on the computation of superconformal indices, see *e.g.* [16], have been discussed in [17, 18]).

Moreover, it has been noted that there is a close relation between superconformal indices in four dimensions [19, 20] and sphere partition functions in three dimensions [21–23]. In this context, the mathematical identities that relate the superconformal indices of dual theories in four dimensions are closely related to the mathematical identities that relate the sphere partition functions of dual theories in three dimensions. This observation raises the possibility of a promising 3D-4D connection.

Finally, it has been proposed [6, 24] that F is a quantity that plays in three dimensions the same role that the c -function plays in two dimensions.¹ In analogy to Zamolodchikov’s c -theorem in two dimensions [27] (or the g -theorem for one-dimensional boundary RG flows [28, 29]), an F -theorem has been conjectured in three dimensions, which states that F decreases along RG flows. Successful checks of this proposal have appeared in [10, 30, 31].

In this paper we will discuss all three of the above aspects in the context of a special class of $\mathcal{N} = 2$ Chern-Simons-Matter (CSM) theories defined as $\mathcal{N} = 2$ Chern-Simons theory at level k coupled to a single chiral superfield in the adjoint representation of the gauge group, which we take to be $U(N)$. We consider this theory in the presence or absence of superpotential interactions. This is an interesting theory for the following reasons:

- (1) In the absence of superpotential deformations the theory is believed to be exactly superconformal for all N, k [32] and exhibits an R-symmetry with large quantum corrections. At strong coupling, *e.g.* large ’t Hooft coupling $\lambda = N/k \gg 1$ in the large- N limit, the R-charge of the adjoint superfield tends to zero [33]. As a result, with increasing coupling more and more fields hit the unitarity bound and decouple from the rest of the theory as free fields, thus truncating the interacting part of the chiral ring

¹The F -theorem was first argued as a general a_d^* -theorem in the context of holography in [25, 26].

from below.

- (2) In the presence of superpotential deformations and sufficiently strong coupling the theory exhibits spontaneous supersymmetry breaking [34]. In the range of parameters where a supersymmetric vacuum exists the superpotential deformation gives rise in the infrared (IR) to a new class of interacting fixed points. These theories are believed to exhibit a Seiberg-like duality that acts on the theory in a self-dual manner [34]. Having a minimal matter content these examples provide some of the simplest illustrations of Seiberg duality in three dimensions and thus a useful playground for attempts to understand the general underpinnings of such dualities in field theory.

Our main goal will be to analyze the implications of these features on the properties of the free energy F . A preliminary analysis of F -maximization in these theories in the large- N 't Hooft limit was presented in [8, 9]. Evaluating F in the saddle point approximation it was confirmed numerically that F -maximization provides at $\lambda \ll N$ (but not necessarily $\lambda \ll 1$) results consistent with a set of non-perturbative inequalities derived in [33]. This is a non-trivial test of the validity of the F -maximization principle. In section 2 we supplement this result with a more thorough examination of the solutions of the saddle point equations. Besides the one-cut solutions of Refs. [8, 9] we demonstrate the existence of a large class of multi-cut solutions and find numerical evidence that the one-cut solutions of Refs. [8, 9] are in fact the dominant ones (namely, they have the smallest free energy in the large- N limit). The multi-cut solutions are expected to play an interesting role in the analysis of the non-perturbative (in $1/N$) instanton contributions to the matrix model that expresses the free energy after localization.

At $\lambda \sim N$ Ref. [8] observed numerically that one of the non-perturbative bounds of [33] gets violated by a naive application of the F -maximization principle. A potential source for this violation is the sizable number of free decoupled fields in this regime. In section 4 we test this expectation by suitably modifying the F -maximization principle to account for the decoupling fields. We find that the modification leads to an R-symmetry consistent with the non-perturbative bounds of [33]. We propose that the implemented modification is the general way to deal with the accidental symmetries of decoupling fields in F -maximization.

Another missing link in the F -maximization prescription is the following. Ref. [2] argues for F -extremization, but the more precise statement about maximization (namely the sign of the second derivative of F with respect to the trial R-charges) is currently mostly an

empirical observation rather than a generally argued fact.² In the one-adjoint theories that we consider we always observe numerically a single maximum that reproduces correctly in appropriate regimes independent information about the exact R-symmetry.

In addition, in section 2 we observe regions in (λ, R) -space (λ being the 't Hooft coupling and R the trial R-charge of the adjoint chiral superfield), where the one-cut solutions cease to exist. This effect occurs precisely when the supersymmetric vacuum of the physical (superpotential-deformed) theory disappears. In the special case of the Chern-Simons matrix model, that expresses the free energy of the topological $\mathcal{N} = 2$ Chern-Simons theory, we demonstrate that this effect is directly related to the vanishing of the full non-perturbative partition function.

This observation motivates the possibility of a *general* relation between zeros of the Q -deformed sphere partition function, which leads to localization, and spontaneous supersymmetry breaking. We formulate and further motivate this potential relation in very general terms in section 3 and appendix B. As a full-fledged theorem this relation would provide a new non-perturbative constraint on spontaneous supersymmetry breaking in three-dimensional QFTs that supplements the known constraints arising from the Witten index.

In section 5 we consider the second theme outlined in the beginning, namely Seiberg duality. In a simple non-topological case, that involves a $U(1)$ and a $U(2)$ CSM theory, we are able to compute the free energies analytically and demonstrate that they match exactly in the dual theories. In more general cases, we find convincing numerical evidence for this matching in the large- N limit using our saddle point solutions.³ In subsection 5.3 further aspects of the duality of interest are discussed in some length.

Finally, the one-adjoint CSM theories possess a large class of RG flows induced by superpotential interactions. In section 6 we outline this web of RG flows and the general predictions implied by the conjectured F-theorem. In subsection 6.3 we find numerical evidence for the validity of these predictions in a set of examples with increasing complexity, thus providing new tests of the conjectured F-theorem.

In two extra appendices (app. A and C) we summarize for the benefit of the reader a set of useful results that are alluded to, but not explicitly analyzed, in the main text.

²Besides the explicit verification of maximization in specific examples, a general relation between F -maximization and volume-minimization has been proposed for large- N CSM gauge theories with candidate Sasaki-Einstein gravity duals [4–6]. See also the more recent work [35].

³Preliminary signs of potential disagreement with Seiberg duality reported in [8] are not verified.

n	3	4	5	6	7	8	9	10
λ_{n+1}^*	0	0.35	0.65	0.9	1.15	1.4	1.7	2

Table 1: The numerically determined values of λ_{n+1}^* using F -maximization [8] for $n = 3, 4, \dots, 10$.

Summary of notation

The one-adjoint CSM theories that we consider are characterized by two integers, N the rank of the $U(N)$ gauge group, and k the level of the CS interaction.⁴ Following [33] we frequently denote the theories without any superpotential interactions as $\hat{\mathbf{A}}$. New interacting fixed points arise by deforming the action with the superpotential interactions $W_{n+1} = \text{Tr } X^{n+1}$, $n = 1, 2, \dots$. We will variably denote the resulting theories either as \mathbf{A}_{n+1} or in more detail as $U(N)_k^{(n+1)}$.

A large- N 't Hooft-like limit is possible and defined as

$$N, k \rightarrow \infty, \quad \lambda = \frac{N}{k} = \text{fixed} . \quad (1.2)$$

In Refs. [8, 33] it was argued that as we increase the coupling λ in the undeformed theory $\hat{\mathbf{A}}$ the R-charge function $R(\lambda)$ decreases monotonically towards zero attaining the values

$$R(\lambda_{n+1}^*) = \frac{2}{n+1}, \quad n = 1, 2, \dots \quad (1.3)$$

at the critical couplings λ_{n+1}^* . At each of these points the operator $\text{Tr } X^{n+1}$ is, by definition, marginal. Using F -maximization, [8, 9], one finds for the first few values of n the values of λ_{n+1}^* listed in Table 1 above.

It has been conjectured that the \mathbf{A}_{n+1} theories exhibit a Seiberg-like duality [34]. In the RG flows that give rise to the \mathbf{A}_{n+1} theories the deforming operator $\text{Tr } X^{n+1}$ is relevant in both the ‘electric’ and ‘magnetic’ versions of the theory when $\lambda \in [\lambda_{n+1}^*, n - \lambda_{n+1}^*]$. Following the nomenclature of analogous situations in four-dimensional gauge theories, we will sometimes call this interval the ‘conformal window’ of the \mathbf{A}_{n+1} theory. A review of the precise details of Seiberg duality in the \mathbf{A}_{n+1} theories appears in section 5.

2. S^3 partition function in the large- N limit

The main object of interest in this paper is the S^3 partition function Z_{S^3} . Using localization techniques [1–3] one can re-express Z_{S^3} as a matrix integral. In our case, this matrix

⁴We assume $k > 0$. The case of negative k can be obtained by a simple parity transformation.

integral reads

$$Z_{S^3}(N, k, R) = \frac{1}{N!} \int \left(\prod_{j=1}^N e^{\mathbf{i}\pi k t_j^2} dt_j \right) \prod_{i < j} (2 \sinh(\pi t_{ij}))^2 \prod_{i,j=1}^N e^{\ell(1-R+\mathbf{i}t_{ij})} = e^{-\mathcal{F}(N,k,R)} \quad (2.1)$$

where t_i are matrix eigenvalues running over the real line and $t_{ij} \equiv t_i - t_j$. $\ell(z)$ is the function

$$\ell(z) = -z \log(1 - e^{2\pi \mathbf{i} z}) + \frac{\mathbf{i}}{2} \left[\pi z^2 + \frac{1}{\pi} \text{Li}_2(e^{2\pi \mathbf{i} z}) \right] - \frac{\mathbf{i}\pi}{12} . \quad (2.2)$$

The free energy that we maximize with respect to R is

$$F = \frac{1}{2}(\mathcal{F} + \bar{\mathcal{F}}) . \quad (2.3)$$

In the \mathbf{A}_{n+1} theories the R-charge is fixed by the superpotential interaction $W_{n+1} = \text{Tr } X^{n+1}$, so in order to compute the free energy F in these theories one simply sets $R = \frac{2}{n+1}$ in the expression (2.1).

2.1. Saddle point equations and their solutions revisited

In the large- N limit the main contribution to Z_{S^3} comes from saddle point configurations that obey the system of algebraic equations

$$\mathcal{I}_i \equiv \frac{\mathbf{i}}{\lambda} t_i + \frac{1}{N} \sum_{j \neq i}^N \left[\coth(\pi t_{ij}) - \frac{(1-R) \sinh(2\pi t_{ij}) + t_{ij} \sin(2\pi R)}{\cosh(2\pi t_{ij}) - \cos(2\pi R)} \right] = 0 , \quad i = 1, 2, \dots, N . \quad (2.4)$$

At a saddle point configuration

$$-\mathcal{F}(\lambda, N) = -\log N! + \sum_{i=1}^N \frac{\mathbf{i}\pi N}{\lambda} t_i^2 + \sum_{i < j}^N \log(4 \sinh^2(\pi t_{ij})) + \sum_{i,j=1}^N \ell(1-R+\mathbf{i}t_{ij}) . \quad (2.5)$$

In general, the N t_i 's that solve these equations are complex numbers (here \mathbb{C} -valued functions of the parameters R, λ).

In lack of an efficient analytic method of solving these equations Refs. [8, 9] proceeded to analyze them numerically. In Ref. [8] solutions were found by introducing a fictitious time coordinate τ (following [36]) and considering the dynamical evolution described by the set of differential equations

$$a \frac{dt_i}{d\tau} = \mathcal{I}_i . \quad (2.6)$$

In these equations $a = e^{\mathbf{i}\pi\theta}$ ($\theta \in [0, 2\pi)$) is a free constant whose choice affects (in some cases) the solution that one converges to. Another crucial input that affects the obtained solution

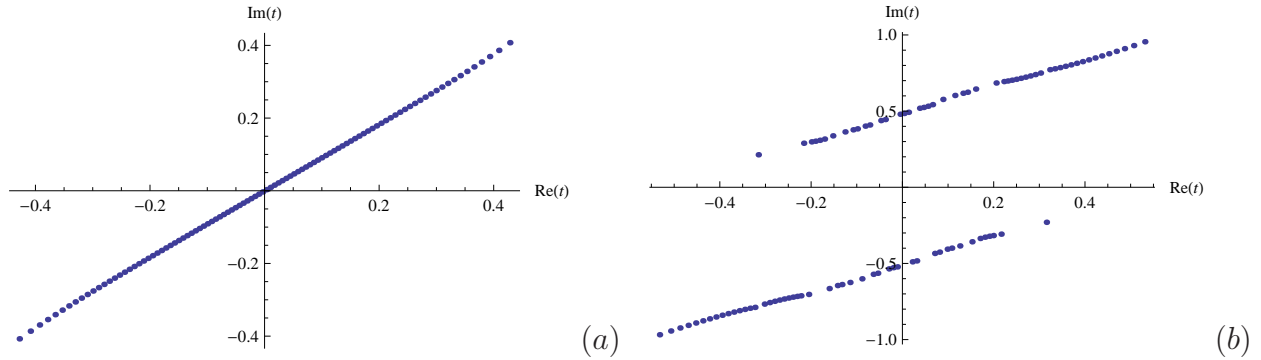


Figure 1: Plot (a): A typical one-cut distribution of the eigenvalues t_i ($i = 1, \dots, N$) in the complex plane. This particular plot has been obtained for $N = 100$, $\lambda = 1$ and $R = 0.28$. Plot (b): A typical two-cut solution depicted here for $N = 100$, $\lambda = 1.5$ and $R = 0.5$. The specific example crosses the imaginary axis at $\pm \frac{i}{2}$.

are the initial conditions at $\tau = 0$. Using two independently written codes in Mathematica and Fortran we searched for different solutions varying a and the initial conditions.

Besides the one-cut solutions that were reported in Refs. [8,9] we have been able to detect many more multi-cut solutions with branches crossing the imaginary axis through the points $\pm \frac{mi}{2}$, $\pm \frac{(R+m)i}{2}$, ($m = 1, 2, \dots$). A typical one-cut solution appears in Fig. 1(a) and a typical two-cut solution in Fig. 1(b). The multi-cut solutions can also be traced analytically in the weak coupling regime ($\lambda \ll 1$) with a perturbative computation (some of the key details of this computation are summarized in appendix A).

The one-cut solutions were the ones used for F -maximization in Refs. [8,9], hence it is important to know if they are also the dominant ones in the large- N limit (*i.e.* if they are the ones having the minimal free energy for fixed λ, R). We confirmed this fact numerically in all cases that we checked. Further credence in the dominance of the one-cut solutions is provided by the following facts: (i) at weak coupling they are the ones that reproduce the perturbative field theory computation of the R -charge [9], (ii) at finite coupling they reproduce to a good accuracy the predictions of Seiberg duality and F-theorem (see below).

The multi-cut solutions are expected to play an interesting role beyond the saddle point approximation. In general, multi-cut solutions are closely related to the non-perturbative instanton corrections that control the large-order behavior of the $1/N$ expansion [37–40]. Since such effects are beyond the immediate scope of the present work in what follows we will concentrate mostly on results based on the one-cut solutions.

2.2. One-cut solutions and the functional behavior of $F(\lambda, R = \frac{2}{n+1})$ in the \mathbf{A}_{n+1} theories

Now we come to a more interesting observation. We focus on the \mathbf{A}_{n+1} theories and fix the R-charge to the value dictated by the deforming superpotential, hence $R = \frac{2}{n+1}$. We would like to understand the functional behavior of the localized free energy function $F(\lambda, \frac{2}{n+1})$ as we vary the 't Hooft coupling λ . In particular, we would like to understand what happens to this function when $\lambda > n$, *i.e.* when the supersymmetric vacuum is lifted. In that case the Q -deformed S^3 partition function computed via localization is no longer equal to the physical partition function of the \mathbf{A}_{n+1} theory, but we can consider it anyway (for example, it gives information about the partition function $Z_{S^3}(\lambda, R)$ of the undeformed $\hat{\mathbf{A}}$ theory before F -maximization at $R = \frac{2}{n+1}$ —see below for more comments related to this aspect).

A particularly instructive example that is amenable to full analytic control is provided by the \mathbf{A}_2 theory, *i.e.* the case with $n = 1$. In that case, the chiral superfield X is massive and the theory flows in the IR to the topological $\mathcal{N} = 2$ Chern-Simons theory. The sphere partition function (2.1) simplifies considerably and can be computed exactly for any N, k . Setting $R = 1$ in (2.1) we obtain (using the identity $\ell(-z) = -\ell(z)$, $z \in \mathbb{C}$)

$$|Z_{S^3}(\lambda, R = 1)| = \frac{1}{N!} \left| \int \left(\prod_{j=1}^N e^{i\pi k t_j^2} dt_j \right) \prod_{i < j}^N (2 \sinh(\pi t_{ij}))^2 \right| = \frac{1}{k^{N/2}} \prod_{m=1}^{N-1} \left(2 \sin \frac{\pi m}{k} \right)^{N-m}. \quad (2.7)$$

For the second equality we used a result that is explained, for example, in appendix B of Ref. [1]. Eq. (2.7) gives the absolute value of the partition function of a well-studied matrix model (the $U(N)$ Chern-Simons matrix model [41])

$$Z_{\text{CS}} = \frac{1}{N!} \int \left(\prod_{j=1}^N e^{-\frac{2\pi^2 N}{S} t_j^2} dt_j \right) \prod_{i < j}^N (2 \sinh(\pi t_{ij}))^2. \quad (2.8)$$

In the conventional case S is a real parameter. In our case $S = 2\pi i \lambda$.

We would like to draw attention to the following property of the expression (2.7): the exact function $Z_{S^3}(\lambda, R = 1)$ vanishes identically in the supersymmetry-breaking regime, $N > k$.⁵ Accordingly, in this regime the free energy F , derived by localization, diverges. Since our information about the general n case is currently limited to the leading-order large- N limit, it is worth elaborating further on how this divergence exhibits itself in the saddle point approximation.

⁵This fact was also observed in Ref. [12].

One way to derive the free energy F at leading order in $1/N$ in the 't Hooft limit is by directly taking the large- N limit of the exact expression (2.7). By converting a sum to an integral we obtain

$$f = -\frac{1}{N^2} \log |Z_{S^3}| = \int_0^1 d\mu (\mu - 1) \log(2 \sin(\pi \lambda \mu)) . \quad (2.9)$$

For $\lambda \leq 1$ this integral evaluates to

$$f = -\frac{\pi i}{12\lambda} (1 - 3\lambda + 2\lambda^2) - \frac{1}{4\pi^2 \lambda^2} \left(\text{Li}_3 \left(e^{-2\pi i \lambda} \right) - \zeta(3) \right) . \quad (2.10)$$

Although not immediately obvious, the rhs of this equation is a real number. For $\lambda > 1$, *i.e.* in the SUSY-breaking regime, part of the integrand in (2.9) lies on a logarithmic branch cut and the expression becomes ill-defined.

Notice that f vanishes at the SUSY boundary value $\lambda = 1$. Hence, the crossover from a well-defined regular f at $\lambda \leq 1$ to a diverging f at $\lambda > 1$ happens discontinuously (more comments on the nature of this discontinuity below). The vanishing of f at the general SUSY boundary values $N = k$ is a property of the exact $|Z_{S^3}|$ for any N , as can be seen from eq. (2.7). This generic feature in the \mathbf{A}_{n+1} theory is a natural property given the vanishing of the rank of the gauge group in the Seiberg dual theory.

Another derivation of eq. (2.9) can be obtained by solving directly the saddle point equations. An analytic solution of these equations is known [42, 43]. For real S in (2.8) the solution is a one-cut solution along the real axis with the eigenvalues occupying the interval

$$\mathcal{C} = [-c, c] , \quad c = \frac{1}{\pi} \text{arccosh} \left(e^{S/2} \right) . \quad (2.11)$$

The resolvent of the solution is

$$\omega(t) = 2 \log \left[e^{-\frac{S}{2}} e^{\pi t} \left(\cosh(\pi t) - i \sqrt{e^S - \cosh^2(\pi t)} \right) \right] , \quad (2.12)$$

and the eigenvalue density

$$\rho(t) = -\frac{1}{\pi} \text{Im} \omega(t) = -\frac{1}{\pi i} \log \frac{\cosh(\pi t) - i \sqrt{e^S - \cosh^2(\pi t)}}{\cosh(\pi t) + i \sqrt{e^S - \cosh^2(\pi t)}} . \quad (2.13)$$

By analytically continuing this result to $S \rightarrow 2\pi i \lambda$ we obtain the one-cut solution that we are interested in. From the endpoint of the cut at

$$c = \frac{1}{\pi} \text{arccosh} \left(e^{\pi i \lambda} \right) \quad (2.14)$$

it is apparent that as we increase λ from 0 to 1, the cut rotates away from the real axis towards the imaginary axis. At $\lambda = 1$ the cut lies fully along the imaginary axis. Including

the complex conjugate configurations that correspond to $\lambda \in [-1, 0]$ one covers the full range of solutions provided by the analytically continued version of (2.12). In other words, there are no one-cut solutions that can be identified in this way for $|\lambda| > 1$, which corresponds to the non-supersymmetric regime of the \mathbf{A}_2 theory.

This picture is fully reproduced by the numerical results of the previous subsection. We reproduce the specifics of the eigenvalue distribution, *e.g.* the λ -dependence of the one-cut endpoint in (2.14), and the functional dependence of the free energy dictated by the expression (2.9) for $\lambda \in [0, 1]$ (see Fig. 2(a)).⁶ When we try to extend the numerical computation above $\lambda = 1$ we fail to find any regular one-cut solutions and are instead able to obtain only multi-cut solutions. In view of the exact result (2.7), it is natural to associate the breakdown of the one-cut solutions at $\lambda > 1$ to the non-perturbative divergence of F .

From a large- N matrix model point of view one anticipates the following picture. In different regions of moduli space the matrix model is dominated by different saddle points. In cases where a one-cut phase dominates the corrections coming from multi-cut phases are exponentially small. In other cases, however, different multi-cut phases are of the same order, they sum in a coherent way and the partition function can have zeroes. This type of asymptotics and phase transitions were studied in Ref. [44]. A more thorough discussion on the vanishing locus of the partition function can be found in the unpublished lecture notes [45].⁷

Having a detailed understanding of the $n = 1$ case, we can now proceed to ask what happens at generic n where our knowledge is unfortunately restricted only to a numerical solution of the large- N saddle point equations.

Our data indicate that the picture is qualitatively similar at all values of n . Nice one-cut solutions exist throughout the whole supersymmetric interval, $\lambda \in [0, n]$, but disappear discontinuously when the supersymmetric vacuum of the \mathbf{A}_{n+1} theory is lifted. Extending the lessons of the $n = 1$ case on the basis of the general picture outlined above, we propose that this breakdown is always due to the non-perturbative divergence of F in the supersymmetry-breaking region.

⁶One can view the good comparison of these data with the analytically expected result (2.9) as an additional test of the accuracy of our numerical results at $N = 100$.

⁷In [45] it is pointed out that zeroes of the partition function appear in many statistical systems as generalization of Lee-Yang zeroes. By the general theory of asymptotic expansions, they condense along anti-Stokes lines and in matrix models they are related to the zeroes of the ϑ -function (see, for example, the footnote in page 39 of Ref. [45]). We are grateful to Marcos Mariño for explaining these aspects to us and for drawing our attention to the Refs. [44, 45].

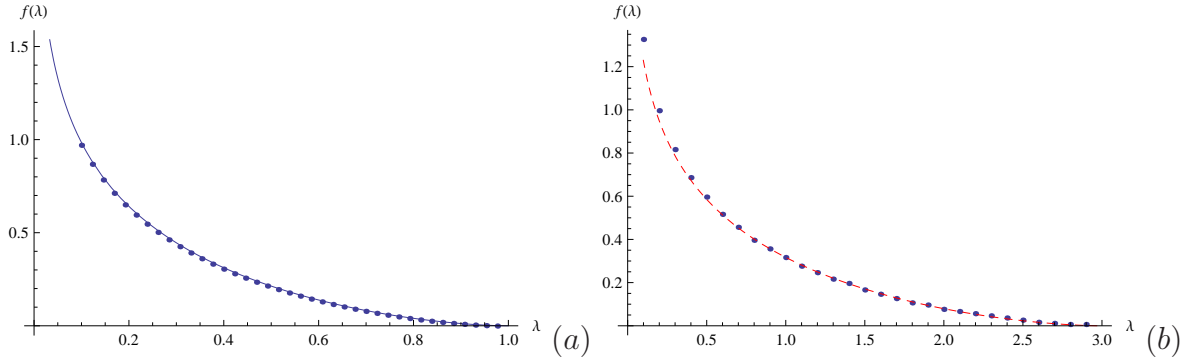


Figure 2: Plot (a) depicts the numerically determined renormalized free energy $f(\lambda)$ for the topological \mathbf{A}_2 theory (blue data points) in the supersymmetric interval $[0, 1]$. The blue curve depicts the analytic result (2.9). Plot (b) provides the analogous numerical result for the \mathbf{A}_4 theory (blue data points) for $\lambda \in [0, 3]$. The dashed red curve is a fit provided by the expression (2.15).

A specific example of the numerically determined λ -dependence of the free energy f in the supersymmetric interval is depicted in Fig. 2(b) for the \mathbf{A}_4 theory. There is no known exact expression for the localized free energy in this non-topological case. Interestingly, we observe that the function

$$f_{fit,n=3} = 0.8 \int_0^1 d\mu (\mu - 1) \log \left(2 \sin \left(\frac{\pi \lambda \mu}{3} \right) \right) \quad (2.15)$$

provides a rather good fit to the numerical data. Qualitatively similar results can be obtained also for other values of n with a fit of the form

$$f_{fit,n} = c_n \int_0^1 d\mu (\mu - 1) \log \left(2 \sin \left(\frac{\pi \lambda \mu}{n} \right) \right) , \quad (2.16)$$

for an appropriately chosen n -dependent number c_n . An appealing feature of this functional form is that it obeys the correct Seiberg duality relation (5.13) that will be discussed later in section 5. Unfortunately, despite this property, (2.16) cannot be the exact expression of the matrix model free energy at leading order in $1/N$ for generic n . At small λ it can be checked that $f(\lambda, R)$ has the expansion detailed in eq. (2.17) below, which would imply according to (2.16) $c_n = 1$ for all n . This would be inconsistent with the presented numerical results.

It remains an interesting open problem to determine analytically the precise expression of f at any n and also to verify non-perturbatively in the matrix model that $Z_{S^3} = 0$ when $N > nk$. As a related point, in Ref. [12] it was shown analytically in a $U(N_c)$ Chern-Simons-Matter theory with N_f pairs of (anti)-fundamental chiral multiplets that the matrix model Z_{S^3} is always vanishing in the supersymmetry-breaking regime $N_c > |k| + N_f$.

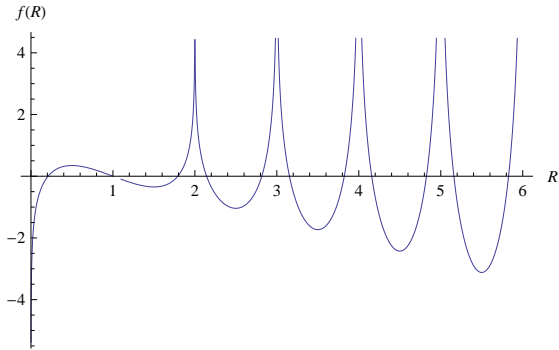


Figure 3: A plot of the free energy of a free field $f(R) = -\ell(1 - R)$.

We conclude this section with a couple of brief comments.

The first comment has to do with the observed discontinuity of the free energy F across the SUSY-breaking boundary. Assuming F (derived by localization) has to diverge in the SUSY-breaking region, the discontinuous jump from a finite value (in our set of examples $F = 0$) to $+\infty$ is consistent with the intuition that the sphere free energy is a natural measure of the number of degrees of freedom. With increasing coupling gauge theories typically exhibit a reduction of the number of degrees of freedom, which is indeed in qualitative agreement with the observation that F is a monotonically decreasing function of λ . In that sense, it would have been puzzling to observe that the free energy increases in the supersymmetric regime near $\lambda = n$ to meet the diverging behavior at $\lambda > n$ continuously.⁸

The second comment is related to the general behavior of the function $F(\lambda, R)$. In the context of F -extremization, it is interesting to know whether F has a single or multiple maxima as a function of R at fixed λ . Numerically we always find a single smooth maximum. More specifically, at small fixed $\lambda \lesssim 0.2$ we observe that $F(R)$ exhibits a behavior similar to that of the free energy of a free field. Within perturbation theory one finds

$$f(\lambda, R) \equiv \frac{1}{N^2} F(\lambda, R) = -\frac{1}{2} \log(4\pi^2 \lambda) - \ell(1 - R) + \mathcal{O}(\lambda^2) . \quad (2.17)$$

The contribution $\ell(1 - R)$ is the free energy of a free field depicted in Fig. 3. As we increase λ the behavior changes to one where the one-cut solutions cease to exist above a critical value of R .⁹ That critical value appears to be set by $R_{crit}(\lambda) = \frac{2}{\lambda+1}$. This is consistent with the previous discussion about the (non)existence of one-cut solutions in the \mathbf{A}_{n+1} theories.

⁸We thank Andrei Parnachev for a question that prompted us to think about this aspect.

⁹We have not been able to determine numerically the precise manner in which this change takes place as we vary λ . We consistently observed a critical value of R for all the values of $\lambda > 0.5$ that we checked.

3. Spontaneous SUSY breaking and zeros of the localized S^3 partition function

The observations of the previous section suggest an intriguing relation between the Q -deformed S^3 partition function that leads to localization and spontaneous supersymmetry breaking, similar in some respects to the one provided by the Witten index [46]. Here we will attempt to formulate and motivate such a relation in a broad descriptive manner postponing a more rigorous treatment to future work.

Consider a general three-dimensional classically superconformal quantum field theory.¹⁰ Place the Euclidean version of this theory on a three-sphere S^3 and compute the partition function

$$Z_{S^3} = \int e^{-S-t \int \{Q,V\}} \quad (3.1)$$

where S is the action of the theory, Q is one of the supercharges and V some fermionic functional. We assume S is a properly defined action that leads to a correct path integral treatment of the theory even at strong coupling. t is the coupling of the Q -exact deformation. V should be chosen in such a way that the deformation $\{Q, V\}$ is positive definite and the standard localization technique can proceed without obstructions [47]. For example, in the case of general Yang-Mills or Chern-Simons gauge theories with arbitrary matter content one can use the prescription of [1–3].

In a theory with a supersymmetric vacuum a standard argument shows that the t -derivative of $Z_{S^3}(t)$ is identically zero. Hence, the physical S^3 partition function defined at $t = 0$ and its infinitely Q -deformed version at $t = \infty$ are identical

$$Z_{S^3}(0) = \lim_{t \rightarrow \infty} Z_{S^3}(t) \equiv Z_{S^3}^{(\text{loc})} . \quad (3.2)$$

In cases where the standard localization techniques [1–3] (based on a UV Lagrangian formulation) can proceed unobstructed and capture correctly the quantum IR physics, the rhs of this equation, $Z_{S^3}^{(\text{loc})}$, can be further expressed in terms of a matrix integral formula. For general $\mathcal{N} = 2$ supersymmetric theories $Z_{S^3}^{(\text{loc})}$ is a function not only of the parameters of the theory (*e.g.* the 't Hooft coupling *etc.*), but also of a set of trial R-charges. F -maximization postulates that the physical values of the R-charges can be determined by minimizing $|Z_{S^3}^{(\text{loc})}|$. So far, this is the familiar story that has led to the recent progress in three-dimensional supersymmetric quantum field theories.

$|Z_{S^3}^{(\text{loc})}|$, computed with this specific procedure, is the quantity one would naturally compute given a specific theory without any a priori knowledge about the fate of supersymmetry

¹⁰For reasons that will become obvious in a moment we assume that the theory has at least $\mathcal{N} = 2$ supersymmetry.

at the quantum level. In a theory with spontaneous supersymmetry breaking the t -derivative of $Z_{S^3}(t)$ is non-zero in general and the equality (3.2) is no longer valid (see appendix B for more details). For that reason the quantity $Z_{S^3}^{(\text{loc})}$ does not compute in this case the physical S^3 partition function of the theory $Z_{S^3}(0)$. Nevertheless, we would still like to claim that $Z_{S^3}^{(\text{loc})}$ is an interesting quantity to consider even when supersymmetry is spontaneously broken.

Our interest in this quantity stems from the potential validity of the following statements in a general classically superconformal quantum field theory (with at least $\mathcal{N} = 2$ supersymmetry):

- (a) $Z_{S^3}^{(\text{loc})} = 0$ implies that the theory exhibits spontaneous supersymmetry breaking.
- (b) Spontaneous supersymmetry breaking implies $Z_{S^3}^{(\text{loc})} = 0$.

Here we refer to spontaneous supersymmetry breaking on S^3 . We believe this is ultimately related to spontaneous supersymmetry breaking on other three-manifolds, *e.g.* $S^2 \times \mathbb{R}$ which is directly related to the information carried by the Witten index [46]. We will not, however, address this connection in this work. Furthermore, since we refer to spontaneous supersymmetry breaking in a finite volume we should mention that this does not necessarily imply that supersymmetry is also broken at infinite volume and the infinite volume limit may need to be treated with care [46].

The first statement is natural for the following reasons. If the theory possesses a supersymmetric vacuum, $Z_{S^3}^{(\text{loc})}$ computes, as argued above, the physical partition function on S^3 . Then, assuming the free energy F is a finite quantity for a quantum field theory with a finite number of degrees of freedom we conclude that $|Z_{S^3}^{(\text{loc})}|$ is a strictly positive quantity. The assumption that F is finite is natural and expected to hold if F defines a sensible analog of the c -function in three dimensions.¹¹ Therefore, the equation $Z_{S^3}^{(\text{loc})} = 0$ cannot be true in a theory with a supersymmetric vacuum and we should infer that, when it vanishes, $Z_{S^3}^{(\text{loc})}$ does not compute the physical partition function $Z_{S^3}(0)$. In that sense, the conclusion of spontaneous supersymmetry breaking seems to be a natural one.

A potential pitfall should be noted here. Currently, the most efficient method to compute $Z_{S^3}^{(\text{loc})}$ proceeds via the localization technique which allows to recast the infinitely Q -deformed path integral (3.1) in terms of a matrix integral. As we pointed out above, this method relies

¹¹In particular, the application of the F -theorem would be subtle in RG flows between theories with infinite F . In such cases $F_{UV} = F_{IR} = \infty$ and the validity of the F -theorem, which states $F_{UV} > F_{IR}$, would be in general unclear.

on the assumption that a certain UV Lagrangian description of the theory captures correctly the quantum IR physics. Incorrect conclusions may be reached if this extra assumption fails. For instance, one could encounter situations where a supersymmetric vacuum exists but the matrix integral computation gives $Z_{S^3}^{(\text{loc})} = 0$. An example of this sort has been presented recently in independent work by Benini, Closset and Cremonesi [48]. The theory in question is three-dimensional super-QCD with gauge group $U(N_c)$ and $N_f = N_c - 1$ pairs of (anti)fundamental multiplets. This theory has a quantum-mechanically deformed moduli space of supersymmetric vacua, but the naive computation of $Z_{S^3}^{(\text{loc})}$ via the standard matrix integral gives $Z_{S^3}^{(\text{loc})} = 0$. Given the fact that the IR theory is a free theory of neutral chiral multiplets, one can argue that the actual value of $Z_{S^3}^{(\text{loc})}$ is non-zero and equal to the value of the undeformed partition function in accordance with our general conjecture. The main lesson is that the sphere partition function Z_{S^3} should be computed always, and at any deformation parameter t including $t = 0$ and $t = \infty$, using the proper formulation of the path integral.

The reverse statement (b) could be motivated (or eventually argued) in a manner outlined in appendix B. Under a set of assumptions outlined in appendix B we argue that there are no states that can contribute non-trivially to the localized path integral $Z_{S^3}^{(\text{loc})}$ when the theory exhibits spontaneous breaking of supersymmetry. This argument is qualitatively similar to the one showing that spontaneous supersymmetry breaking requires that the Witten index vanishes. As in the case of the Witten index, here also we would have been unable to reverse this argument to conclusively argue that $Z_{S^3}^{(\text{loc})} = 0$ implies spontaneous supersymmetry breaking (namely statement (a)). It is interesting that, unlike the Witten index case, for $Z_{S^3}^{(\text{loc})}$ we can find an independent motivation for (a) that goes through the finiteness of F as a rough measure of degrees of freedom and its relation to the F -theorem.

If correct, the combined statements (a) and (b) provide a new powerful non-perturbative constraint on spontaneous supersymmetry breaking in three dimensions. Under the assumption that the standard localization techniques [1–3] proceed unobstructed, they postulate that the free energy of a matrix model determines whether a corresponding 3D quantum field theory does or does not exhibit spontaneous supersymmetry breaking. The examples studied in this paper (and the examples in [12]) provide some evidence in favor of such constraints. The study of $Z_{S^3}^{(\text{loc})}$ in more examples with known patterns of spontaneous supersymmetry breaking may provide further insights into the fate of these statements. We hope to return to this task and a more formal argument in favor of (a) and (b) in a different publication.

Finally, we may add the following potentially interesting aspect to the above story. In the context of statements (a) and (b) it may also be interesting to ask whether $Z_{S^3}^{(\text{loc})}$ combined with the 3D-4D connection, provides a way to relate supersymmetry breaking patterns in three and four dimensions. Similarities in such patterns are observed, for example, between the four-dimensional $\mathcal{N} = 1$ SQCD theory and three-dimensional $\mathcal{N} = 2$ CSM theories with fundamental and antifundamental matter, the four-dimensional $\mathcal{N} = 1$ one-adjoint SQCD theories and $\mathcal{N} = 2$ CSM theories of the type considered in this paper with additional matter in the (anti)fundamental representations *etc.* [33].

4. Decoupling fields and modifications of the F -maximization principle

It can be argued [33] that the exact $U(1)$ R-charge R of the adjoint chiral superfield X in the $\hat{\mathbf{A}}$ theory tends to zero at strong coupling. For example, in the large 't Hooft coupling limit $\lim_{\lambda \rightarrow \infty} R(\lambda) = 0$. In [8, 9] it was indeed shown with F -maximization that R is a monotonically decreasing function of λ that seems to asymptote to zero.

As the R-charge decreases towards zero the scaling dimensions $\Delta_{n+1} = (n+1)R$ of the chiral ring operators $\text{Tr } X^{n+1}$ also decrease and, accordingly, with increasing coupling more and more chiral ring operators hit the unitarity bound and decouple as free fields. It is easy to show [33] that the general operator $\text{Tr } X^{n+1}$ hits the unitarity bound for the first time when the operator $\text{Tr } X^{4(n+1)}$ becomes marginal. When a field decouples, a new accidental symmetry occurs and the exact $U(1)$ R-symmetry of the theory can mix with it. In general, this affects the validity and formulation of the F -maximization principle.

In the large- N , finite $\lambda \ll N$ computations of Refs. [8, 9] these effects were not important, because only a small finite number of free fields existed which had a negligible contribution to the total free energy of the theory that scales like N^2 . However, as one increases λ and makes it of the same order as N the increased number of decoupled fields can have a sizable contribution to the free energy and the F -maximization principle must either be dropped or appropriately modified. The computation of these effects requires a direct analysis of the saddle point equations in the ‘M-theory’ limit — $N \rightarrow \infty$, k finite— which is unfortunately not a straightforward exercise. In the language of Ref. [6] this difficulty is due to the non-cancellation of long-range forces on the eigenvalues. On a more superficial level, Ref. [8] observed numerically that the R-charge obtained by naive F -maximization at $\lambda \sim N = 100$ violated a non-perturbative bound on the R-charge that follows from Seiberg duality and the assumption that R is a monotonically decreasing function of λ . A potential source of

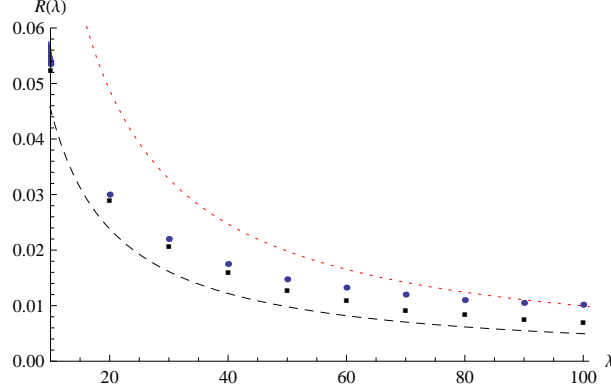


Figure 4: *The numerically determined R -charge curve in the very strong coupling regime for $10 < \lambda < 100$ and $N = 100$. The upper blue points represent the result of the standard F -maximization recipe. The lower black square points represent the result of the modified F -maximization recipe (4.2) which is designed to take into account the effects of the decoupling fields. The exact result is required by consistency to lie within the bounds of the lower dashed black curve $\frac{1}{2(\lambda+1)}$ and the upper dotted red curve $\frac{2}{2\lambda+1}$ (the lower bound holds for $\lambda \in \mathbb{N}$ and the upper bound for $\lambda \in \mathbb{N}/2$) [8].*

this discrepancy is the considerable number of decoupling fields.

The general question of interest here is the following: is there a suitable modification of the F -maximization principle that takes properly into account the effects of decoupling fields in a theory with large anomalous dimensions? In corresponding situations in four dimensions one is instructed to maximize a modified a -function where the 't Hooft anomalies associated to the free fields have been subtracted [49, 50]. In this section we want to consider and test an analogous modification for F -maximization in three dimensions.

A natural course of action would simply be to subtract the free energy of the decoupling fields. If \mathcal{O} is a decoupled operator with trial scaling dimension Δ its contribution to the free energy is

$$F_{free} = -\ell(1 - \Delta) . \quad (4.1)$$

Therefore, if m chiral operators, $\text{Tr } X, \text{Tr } X^2, \dots, \text{Tr } X^m$ have decoupled in our theory at 't Hooft coupling λ , we propose that one should not maximize the free energy $F(\lambda, R)$ that follows from the matrix integral (2.1), but the modified free energy

$$F_{\text{mod}}(\lambda, R; m) = F(\lambda, R) + \sum_{i=1}^m \ell(1 - mR) . \quad (4.2)$$

We implemented numerically this modified F -maximization principle to the $\hat{\mathbf{A}}$ theory and the result is plotted (for $N = 100$) in Fig. 4. At each λ we determined m , the number

of free fields, in a self-consistent manner by requiring $\text{Tr } X^m$ to be the highest in polynomial power chiral ring operator that is decoupled after F_{mod} -maximization is implemented. In Fig. 4 the blue round dots are the result of the standard F -maximization procedure that exhibits the above-mentioned violation near $\lambda \sim 100$ (*i.e.*, the curve traced by the blue data points crosses the upper-bounding red curve given by $\frac{2}{2\lambda+1}$) [8]. The black square dots are the result of F_{mod} -maximization. We observe that the modified principle brings down the value of R towards the black dashed lower-bound curve. The new R-charge curve lies between the upper and lower bound curves and exhibits no signs of bound violation.¹² Moreover, a naive fit of the black data points gives at large λ the following asymptotic behavior of $R(\lambda)$

$$R(\lambda) \sim 0.46 \lambda^{-0.91} \quad (4.3)$$

which agrees with the intuition that $R(\lambda)$ should trace closely (or asymptote to) the lower bounding curve given by $\frac{1}{2(1+\lambda)}$ [8].

This short computation is suggestive of the validity of F_{mod} -maximization, but unfortunately does not provide conclusive evidence. The above numerical computation has been performed at a finite, but large, value of N where the leading order saddle point approximation is corrected by several sources of $1/N$ contributions. It is not completely clear how these $1/N$ effects compare with the contributions due to the decoupling fields. For that reason, it would be desirable to implement and test our proposal in more examples of CSM theories that exhibit decoupling fields, *e.g.* examples that include extra fields in (anti)fundamental representations [33]. In such cases, the contribution of decoupling meson-like operators would be of the same order as the leading order free energy.

5. Duality in \mathbf{A}_{n+1} theories

In Ref. [34] evidence was provided for a Seiberg-like duality in \mathbf{A}_{n+1} theories.¹³ At finite N , k the duality relates the $U(N)$ \mathbf{A}_{n+1} theory at level k to the $U(nk - N)$ theory at level $-k$; in convenient notation

$$U(N)_k^{(n+1)} \leftrightarrow U(nk - N)_{-k}^{(n+1)}. \quad (5.1)$$

¹²For a clear explanation of the origin and meaning of the bound curves we refer the reader to [8, 33].

¹³Ref. [33] considered generalizations of this duality in theories with one or two adjoint chiral superfields and additional chiral superfields in the (anti)fundamental representations. In Ref. [51] a duality was proposed for an $SU(2)$ $\mathcal{N} = 2$ CS theory at level one coupled to a single chiral superfield in the adjoint and no superpotential interactions. A Seiberg duality in three dimensional Chern-Simons-Matter theories was originally proposed by Gaiotto and Witten in [52] for a theory with matter in (anti)fundamental representations.

In this case Seiberg duality acts self-dually and exchanges only the rank of the gauge group and the sign of the CS level. In the large- N 't Hooft limit the duality acts by exchanging $\lambda \leftrightarrow n - \lambda$ (where here we define $\lambda = \frac{N}{|k|}$). In terms of the number of matter fields involved this is one of the simplest known dualities in three dimensions.

Let us briefly review the arguments in favor of this duality. Originally, this duality was motivated in Ref. [34] with a standard D-brane argument based on a type IIB string theory setup that involves N D3-branes suspended between n NS5-branes and a single $(1, k)$ 5-brane bound state suitably oriented to preserve the right amount of supersymmetry. In order to argue for duality one moves the fivebranes through each other and considers the effects of this motion on the D3-branes (which is the place where the Chern-Simons-Matter theory of interest lives). To the degree that this brane motion is inconsequential for the IR physics, this is an argument in favor of duality in the IR field theory on the D3-branes. A potentially subtle point in this procedure has to do with the fact that, because of the particular orientation of the branes, the fivebranes have to meet and cross each other in spacetime at a point where singular behavior can arise triggering a transition. Such phenomena are known to occur, but the presence of the Chern-Simons interaction is believed to alleviate the singular behavior and help the theory avoid a transition (see [12] for a relevant discussion and explicit examples).

In our case, additional evidence for the validity of the duality is provided by the following facts:

- (1) In the special case where $n = 1$ the theory flows in the IR to the topological $\mathcal{N} = 2$ CS theory. Duality in this case reduces to level-rank duality and thus one can check explicitly the matching of the S^3 partition functions [12].
- (2) In the general n case, we can further deform the superpotential to

$$W = \sum_{i=0}^n \frac{g_i}{n+1-i} \text{Tr } X^{n+1-i} \quad (5.2)$$

and flow in the deep IR to a product of \mathbf{A}_{n_i} theories ($n_i < n$). Suitably arranging the coefficients g_i one can flow to a product of \mathbf{A}_2 theories

$$U(N)_k^{(n+1)} \rightarrow U(N_1)_k^{(2)} \times \cdots \times U(N_n)_k^{(2)}, \quad \sum_{i=1}^n N_i = N. \quad (5.3)$$

As already noted, the duality holds in each of the $U(N_i)_k^{(2)}$ factors as level-rank duality. One might expect that the duality continues to hold as we tune back all the coefficients

g_i ($i = 1, \dots, n$) to zero, keeping g_0 non-zero, in order to recover the original $U(N)_k^{(n+1)}$ theory. A more detailed discussion of these RG flows will appear in the next section 6. The behavior of the duality under the general superpotential deformation is further discussed in appendix C.

In what follows we attempt to find more detailed tests of the duality using the localized S^3 partition function. The duality predicts that the S^3 partition function is invariant under duality (up to an overall phase which is generally attributed to different framing [12, 53]). In our case, this statement translates to the following set of mathematical identities (for $N \leq nk$)

$$\begin{aligned} Z_{S^3} \left[U(N)_k^{(n+1)} \right] &= \frac{1}{N!} \int \left(\prod_{j=1}^N e^{i\pi k t_j^2} dt_j \right) \prod_{i < j}^N (2 \sinh(\pi t_{ij}))^2 \prod_{i,j=1}^N e^{\ell(\frac{n-1}{n+1} + i t_{ij})} = \\ e^{i\vartheta(N,k,n)} Z_{S^3} \left[U(nk - N)_{-k}^{(n+1)} \right] &= \\ \frac{e^{i\vartheta(N,k,n)}}{(nk - N)!} \int \left(\prod_{j=1}^{nk-N} e^{-i\pi k t_j^2} dt_j \right) \prod_{i < j}^{nk-N} (2 \sinh(\pi t_{ij}))^2 \prod_{i,j=1}^{nk-N} e^{\ell(\frac{n-1}{n+1} + i t_{ij})} \end{aligned} \quad (5.4)$$

where $\vartheta(N, k, n)$ is some phase. In other words, the free energies F are duality invariant. We will test the validity of these identities first exactly in a simple, but non-topological, low- N case, and then in the large- N 't Hooft limit by using the results of our numerical computation. The matching in the latter case also provides supplemental evidence for the validity of these results and the accuracy of the numerical computation.

Another way to derive the duality invariance of the free energy is by reduction of a superconformal index in four dimensions [21]. Starting from the electric-magnetic duality of an $Sp(2N)$ theory in [54] it is possible to reduce the matching of the corresponding superconformal indices to the equality of three-dimensional partition functions in (5.4).¹⁴

Other detailed aspects of the duality will be discussed in subsection 5.3.

5.1. A simple non-topological duality

One can readily check the validity of the identities (5.4) in the topological $n = 1$ case. The ‘hard’ $e^{\ell(\dots)}$ factors drop out in this case and the integrals can be computed straightforwardly by brute force (see *e.g.* eq. (2.7)). In general, one finds [12]

$$Z_{S^3} \left[U(N)_k^{(2)} \right] = e^{-\frac{\pi i}{12}(k^2 - 6k + 2)} Z_{S^3} \left[U(k - N)_{-k}^{(2)} \right]. \quad (5.5)$$

¹⁴We thank Grigory Vartanov for a discussion on this aspect.

Now let us consider a simple non-topological case. We set $N = 2, k = 1, n = 3$ which leads to the duality

$$U(2)_1^{(4)} \leftrightarrow U(1)_{-1}^{(4)} . \quad (5.6)$$

The $U(1)$ partition function involves a single Gaussian integral that gives (by analytic continuation)

$$Z_{S^3} [U(1)_{-1}^{(4)}] = \frac{1}{\sqrt{2}} e^{-\frac{\pi i}{4}} . \quad (5.7)$$

The $U(2)$ partition function involves a double integral

$$Z_{S^3} [U(2)_1^{(4)}] = \frac{1}{2} \int_{-\infty}^{\infty} dt_1 dt_2 e^{i\pi(t_1^2+t_2^2)} \frac{\sinh^2(\pi t_{12})}{\cosh(\pi t_{12})} . \quad (5.8)$$

We used the fact that $e^{2\ell(\frac{1}{2})} = \frac{1}{2}$. With a simple change of integration variables we can factor out a Gaussian integral and reduce the computation to

$$Z_{S^3} [U(2)_1^{(4)}] = \frac{1}{2\sqrt{2}} e^{-\frac{3\pi i}{4}} \int_{-\infty}^{\infty} dt e^{\frac{\pi i}{2} t^2} \frac{\sinh^2(\pi t)}{\cosh(\pi t)} . \quad (5.9)$$

Using the identity $\sinh^2(\pi t) = \cosh^2(\pi t) - 1$ we can further split this integral into a Gaussian integral and an integral that can be computed directly with Mathematica

$$Z_{S^3} [U(2)_1^{(4)}] = \frac{1}{2\sqrt{2}} e^{-\frac{3\pi i}{4}} \left(\sqrt{2} e^{\frac{3\pi i}{4}} - \left(2e^{-\frac{\pi i}{8}} - \sqrt{2} e^{-\frac{\pi i}{4}} \right) \right) = \frac{1}{\sqrt{2}} e^{\frac{\pi i}{8}} . \quad (5.10)$$

We conclude that

$$Z_{S^3} [U(2)_1^{(4)}] = e^{\frac{3\pi i}{8}} Z_{S^3} [U(1)_{-1}^{(4)}] \quad (5.11)$$

which verifies the prediction made by the duality in this case.

Notice that the relative phase in (5.11) is not the same as that in eq. (5.5) for $k = 1$. This implies that the relative phase has, in general, both a k and an n dependence.

The particular example we have just discussed demonstrates the important role of the $U(1)$ part of the gauge group in the duality. For an $SU(2)$ theory at level 1 and no superpotential interaction a different type of duality was conjectured in [51].

5.2. Free energy matching tests at large- N

We can provide additional tests of the above duality in the large- N 't Hooft limit by using the numerical results of section 2. Focusing on the planar part of the free energy we define

$$f(\lambda) = \frac{1}{N^2} F(\lambda, N) . \quad (5.12)$$

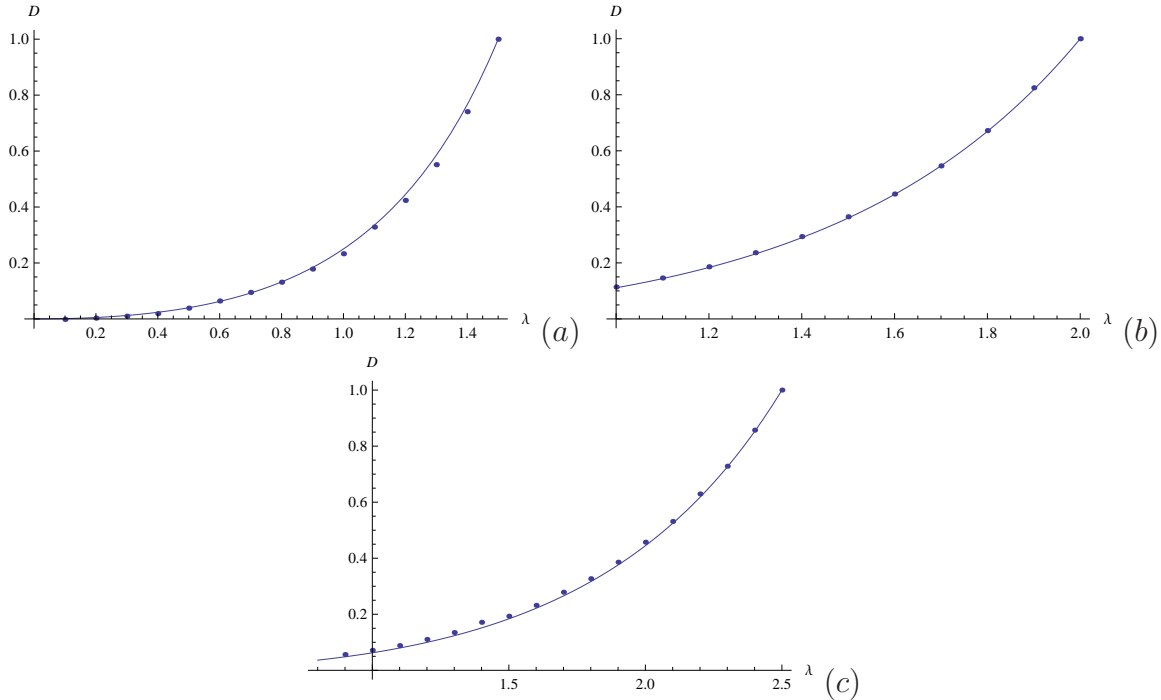


Figure 5: Plots of the function $D(\lambda)$, defined in terms of the free energies in eq. (5.14), for the \mathbf{A}_4 theory (plot (a)), the \mathbf{A}_5 theory (plot (b)) and the \mathbf{A}_6 theory (plot (c)). The blue points represent the numerically computed results and the solid curves the duality-predicted functions $\lambda^2(n - \lambda)^{-2}$.

The duality predicts the equality

$$F(\lambda, N) = F\left(n - \lambda, \left(\frac{n}{\lambda} - 1\right) N\right) \Leftrightarrow f(\lambda) = \left(\frac{n}{\lambda} - 1\right)^2 f(n - \lambda) . \quad (5.13)$$

In Fig. 5 we define

$$D(\lambda) \equiv \frac{f(n - \lambda)}{f(\lambda)} \quad (5.14)$$

which according to the duality should be equal to $\lambda^2(n - \lambda)^{-2}$. Using our numerical data we present how the matching works in three cases: $n = 3, 4, 5$ (*i.e.* the $\mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6$ theories). For $n = 3$ the conformal window extends over the whole supersymmetric interval $\lambda \in [0, 3]$. For $n = 4$ the conformal window extends over the range $\lambda \in (0.35, 3.65)$. For $n = 5$ the conformal window extends over the range $\lambda \in (0.65, 4.35)$. In all cases we cover most of the conformal window fixing the probed range according to the ability of the numerical code to produce trustworthy results. The plotted data terminate at the self-dual point of each theory which lies at $\lambda_{sd} = \frac{n}{2}$. The agreement between the curve traced by the numerical points and the curve $\lambda^2(n - \lambda)^{-2}$ predicted by duality is rather convincing.

5.3. Other aspects of duality

The successful matching of the free energies F in the above examples, combined with the arguments summarized in the beginning of this section, provide non-trivial evidence for the validity of the dualities (5.1). It would be very interesting to show analytically the matching of the free energies (5.4) for general N, k (modulo potential intricacies that are discussed below). Transformation properties of hyperbolic hypergeometric integrals [55–58] may be very useful in this respect (as they have been in other cases, *e.g.* in the case of the Giveon-Kutasov duality [13, 52]). To the best of our knowledge the identities (5.4) have not been proven in this way before (see [59] for the appearance of analogous identities in four-dimensional theories). In anticipation of such results we would like to offer at this point a few additional comments on some particularly interesting, less discussed, physical aspects of the dualities (5.1). For simplicity, we will focus on the large- N 't Hooft limit where the \mathbf{A}_{n+1} theories are parameterized by n and the continuous 't Hooft coupling λ .

We have argued (see [33] for a detailed discussion) that there is a critical coupling λ_{n+1}^* where the deforming operator $\text{Tr } X^{n+1}$ becomes marginal. For $\lambda < \lambda_{n+1}^*$ the deforming operator is irrelevant (for $n > 3$) and the theory erases the deformation in the deep IR where one recovers the undeformed $\hat{\mathbf{A}}$ theory. For $\lambda > \lambda_{n+1}^*$ the deforming operator is relevant and drives the theory to a new IR fixed point, which is the theory we are interested in. The global symmetry group of this theory is a single $U(1)$, the R-symmetry group. The R-charge of the adjoint chiral superfield is fixed by the superpotential

$$R(X) = \frac{2}{n+1} . \quad (5.15)$$

The duality is expected to match symmetries and operators. Indeed, this works very well inside the range of λ 's that we call the conformal window of this theory. This is the range where the deforming operator $\text{Tr } X^{n+1}$ is relevant both in the 'electric' $U(N)$ theory and the 'magnetic' $U(nk - N)$ theory, namely when

$$\lambda \in [\lambda_{n+1}^*, n - \lambda_{n+1}^*] . \quad (5.16)$$

In this case, the $U(1)$ R-symmetry is identically the same on both sides of the duality because it is controlled by the same superpotential interaction, *i.e.*

$$R(X) = R(\bar{X}) = \frac{2}{n+1} \quad (5.17)$$

where \bar{X} denotes the adjoint chiral superfield on the magnetic side. Notice, that because of this, the number of decoupled free chiral operators is the same on both sides of the duality.

Specifically, the single-trace operators

$$\mathrm{Tr} X, \mathrm{Tr} X^2, \dots, \mathrm{Tr} X^{\lfloor \frac{n+1}{4} \rfloor} \quad (5.18)$$

are free and decoupled on both sides of the duality ($[x]$ denotes the integer part of the number x). Accordingly, under duality not only F but also its modified version (4.2) is invariant.

As an aside comment, we note the following interesting property of the RG flow from the UV undeformed $\hat{\mathbf{A}}$ theory to the IR deformed theory \mathbf{A}_{n+1} in this range of λ . In the UV the number of free decoupled fields is m and this number is controlled by the exact R-symmetry of the $\hat{\mathbf{A}}$ theory which is determined by F -maximization. Specifically,

$$m = \left\lfloor \frac{1}{2R(\lambda)} \right\rfloor. \quad (5.19)$$

Along the RG flow from $\hat{\mathbf{A}}$ to \mathbf{A}_{n+1} the R-charge increases and m drops in general from (5.19) to $\left\lfloor \frac{n+1}{4} \right\rfloor$, which implies that a number of UV-free operators recouple in the IR and regain their interacting status. It would be useful to obtain a deeper understanding of the mechanism that realizes this effect.

Returning to the \mathbf{A}_{n+1} theory one can do a bit better in matching the operators. Following [60] in appendix C we demonstrate how one matches the general superpotential deformation (5.2) and the associated vacuum structure on both sides of the duality. A simple corollary of that analysis is the map

$$\mathrm{Tr} X^i \leftrightarrow -\mathrm{Tr} \bar{X}^i, \quad i = 1, 2, \dots, n+1 \quad (5.20)$$

for electric superpotential $W = \mathrm{Tr} X^{n+1}$ and magnetic superpotential $\bar{W} = \mathrm{Tr} \bar{X}^{n+1}$. Under this map, a free decoupled operator from the list (5.18) on the electric side maps to a free decoupled operator from the corresponding list on the magnetic side.

This appears to be a satisfactory picture of how 3D Seiberg duality works in this class of Chern-Simons-Matter theories. Now one can ask what happens when λ is increased further and dialed to lie in the window

$$[n - \lambda_{n+1}^*, n] \quad (5.21)$$

where a supersymmetric vacuum continues to exist, but the deforming operator is relevant in the electric theory and *irrelevant* in the magnetic theory. F -maximization has allowed us to check that the size of this window is non-zero (this is essentially the statement that the exact R-charge curve, determined by F -maximization, lies below the upper bounding red curve in Fig. 4). This regime is analogous to the regime of the free magnetic phase in four-dimensional

$\mathcal{N} = 1$ SQCD. The standard four-dimensional Seiberg duality is expected to work also in this phase by exchanging the ‘very-strongly’ coupled electric description to the IR free magnetic description. In our three-dimensional example we find a similar effect. In the vicinity of the supersymmetry breaking point, $0 < n - \lambda \ll 1$, the superpotential deformation is so strongly coupled that it is in the verge of lifting the supersymmetric vacuum. In the dual description, where the dual ’t Hooft coupling is small, $\bar{\lambda} = n - \lambda \ll 1$, the superpotential deformation is irrelevant. In this case the mismatching of symmetries, number of free decoupled operators *etc.* suggests that the correct R-symmetry cannot be read off the deforming superpotential in the electric description, and that one should rather use the magnetic description to study the theory in this regime (see [50] for analogous comments in the case of the four-dimensional one-adjoint SQCD theory).

It is interesting to understand how this story implements itself on the level of the free energies F computed via localization in the regime (5.21). A naive possibility that seems unlikely to work (for the above reasons) is to try to match the free energy computed via localization at $R = \frac{2}{n+1}$ in the electric theory and the free energy computed via localization at R determined by F -maximization in the $\hat{\mathbf{A}}$ theory on the magnetic side. Another possibility is that the mathematical identities that give rise to (5.4) are blind to the existence of a conformal window and apply for all N, k in the supersymmetric interval. In that case, outside the conformal window the matrix model F does not refer to the actual free energy of the theory. Yet another possibility is that they only apply for N, k within the conformal window, but then it is interesting mathematically to identify precisely how they break down outside of it.

6. Tests of the F-theorem at large- N

Recently, an analog of the c -theorem in two dimensions has been put forward for three-dimensional quantum field theories in Refs. [6, 24]. The precise proposal, which is known as the F-theorem, states that F (the free energy of the S^3 partition function we have been computing) always decreases along RG flows. In the absence of a general proof it is instructive to extend the tests in as many classes of theories as possible. In this section we test the validity of the theorem in a new set of RG flows that occur within the setup of the $\hat{\mathbf{A}}$ and \mathbf{A}_{n+1} theories.

6.1. A web of RG flows

By adding relevant superpotential interactions to the one-adjoint $\hat{\mathbf{A}}$ CSM theory one can generate a web of RG flows. We have already discussed some of these flows. In the range of coupling where a chiral ring operator $\text{Tr } X^{n+1}$ is relevant (there is such a range for any $n > 1$), one can use the operator to deform the $\hat{\mathbf{A}}$ theory by the corresponding superpotential and generate a supersymmetric RG flow towards a new IR fixed point that we called \mathbf{A}_{n+1} .

Another general class of RG flows arises by using an \mathbf{A}_{n+1} theory as the UV fixed point and deforming it by a general polynomial superpotential of the form

$$W = \sum_{i=0}^n \frac{g_i}{n+1-i} \text{Tr } X^{n+1-i} . \quad (6.1)$$

The supersymmetric vacua of this theory can be found by solving the F -term equations

$$W'(x) = \sum_{i=0}^n g_i x^{n-i} = g_0 \prod_{i=1}^{\ell} (x - a_i)^{n_i} , \quad \sum_{i=1}^{\ell} n_i = n \quad (6.2)$$

where a_i are parameters directly related to the coefficients g_i of the superpotential polynomial. In the far infrared the theory flows to a product of \mathbf{A}_{n_i+1} theories specified by a partition of the total rank N

$$U(N)_k^{(n+1)} \rightarrow U(N_1)_k^{(n_1+1)} \times \cdots \times U(N_{\ell})_k^{(n_{\ell}+1)} , \quad \sum_{i=1}^{\ell} N_i = N . \quad (6.3)$$

There is a supersymmetric vacuum for each of the factors in this product provided $N_i \leq n_i k$ for all $i = 1, 2, \dots, \ell$.

6.2. General predictions of the conjectured F -theorem

For each of the above RG flows the F -theorem predicts the inequality

$$F_{UV} > F_{IR} . \quad (6.4)$$

For the general flow (6.3) this inequality implies

$$F_{N,k}^{(n+1)} > \sum_{i=1}^{\ell} F_{N_i,k}^{(n_i+1)} . \quad (6.5)$$

Focusing on the planar contributions $f(\lambda; n)$ in the large- N limit (see eq. (5.12)) we obtain

$$f\left(\sum_{i=1}^{\ell} \lambda_i; \sum_{i=1}^{\ell} n_i\right) > \sum_{i=1}^{\ell} x_i^2 f(\lambda_i; n_i) \quad (6.6)$$

where by definition

$$N_i = x_i N, \quad \lambda_i = x_i \lambda, \quad 0 < x_i < 1, \quad \sum_{i=1}^{\ell} x_i = 1. \quad (6.7)$$

Notice that in these flows it is in general possible to start in the UV from an \mathbf{A}_{n+1} theory inside its conformal window and end with a product of theories where some of them lie outside their conformal window. In these theories the deforming operator may be irrelevant in which case the true IR fixed point is the $\hat{\mathbf{A}}$ theory and the free energy should be computed accordingly.

6.3. Tests of the F -theorem

We proceed to demonstrate the validity of the above inequalities in a set of examples with increasing complexity. We will concentrate on the large- N limit where we can make use of the numerical results of section 2.

6.3.1. $\hat{\mathbf{A}} \rightarrow \mathbf{A}_{n+1}, \mathbf{A}_{n+1} \rightarrow \mathbf{A}_{n'+1}, n' < n$

We begin with the RG flow $\hat{\mathbf{A}} \rightarrow \mathbf{A}_{n+1}$. We pick $\lambda < n$ to be such that the deforming operator $\text{Tr } X^{n+1}$ is relevant in the UV theory. In these flows

$$R_{UV} < R_{IR} = \frac{2}{n+1}. \quad (6.8)$$

Since $F(R_{UV})$ (determined by F -maximization) is a global maximum in all these cases, the inequality (6.4) immediately follows.

As a simple example of the general flow (6.3) we can consider the case

$$U(N)_k^{n_{UV}+1} \rightarrow U(N)_k^{n_{IR}+1}, \quad n_{IR} < n_{UV}, \quad N \leq n_{IR}k. \quad (6.9)$$

In this case, the 't Hooft coupling λ is constant along the RG flow. We choose it so that it lies inside the conformal window of the UV theory, namely $\lambda \in [\lambda_{n_{UV}+1}^*, n_{UV} - \lambda_{n_{UV}+1}^*]$. Since $\lambda_{n_{IR}+1}^* < \lambda_{n_{UV}+1}^*$, it follows that the IR theory will also be within its conformal window. As a result,

$$R_{UV} = \frac{2}{n_{UV}+1} < R_{IR} = \frac{2}{n_{IR}+1}. \quad (6.10)$$

The monotonicity of F as a function of R (at fixed λ) guarantees again the F -theorem inequality (6.4).

6.3.2. $\mathbf{A}_8 \rightarrow \mathbf{A}_4 \otimes \mathbf{A}_5$

We move on to a more intricate one-parameter family of examples

$$(n = 7, \lambda) \rightarrow (n = 3, \lambda x) \otimes (n = 4, \lambda(1 - x)) , \quad 0 < x < 1 . \quad (6.11)$$

Further constraints on the parameter x may arise by requiring that the IR theories have a supersymmetric vacuum. Being obvious we will not write out these constraints explicitly below (in the specific cases that we consider they are in any case vacuous). The range of the conformal windows of the $\mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_8$ theories are

$$\mathbf{A}_4 : (0, 3) \quad , \quad \mathbf{A}_5 : (0.35, 3.65) \quad , \quad \mathbf{A}_8 : (1.15, 5.85) . \quad (6.12)$$

For illustration purposes let us take $\lambda = 2$ (similar results can be obtained also for other values of λ). The F-theorem inequality (6.6) becomes in this case

$$\Delta(x) = f(2; 7) - x^2 f(2x; 3) - (1 - x)^2 f(2(1 - x); 4) > 0 . \quad (6.13)$$

The results of the numerical computation are depicted in plot (a) of Fig. 6. We observe that the inequality (6.13) is verified.

6.3.3. $\mathbf{A}_{10} \rightarrow \mathbf{A}_6 \otimes \mathbf{A}_5$

The results of a similar computation with different values of n are depicted in plot (b) of Fig. 6. Now we consider the one-parameter family of flows

$$(n = 9, \lambda) \rightarrow (n = 5, \lambda x) \otimes (n = 4, \lambda(1 - x)) , \quad 0 < x < 1 \quad (6.14)$$

and for concreteness we set $\lambda = 3$. The range of the conformal windows of the $\mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_{10}$ theories are

$$\mathbf{A}_5 : (0, 35, 3.65) \quad , \quad \mathbf{A}_6 : (0.65, 4.35) \quad , \quad \mathbf{A}_{10} : (1.7, 7.3) . \quad (6.15)$$

The F-theorem inequality (6.6) becomes in this case

$$\Delta(x) = f(3; 9) - x^2 f(3x; 5) - (1 - x)^2 f(3(1 - x); 4) > 0 . \quad (6.16)$$

Again, it is apparent from the results depicted in plot (b) of Fig. 6 that the inequality (6.19) is verified.

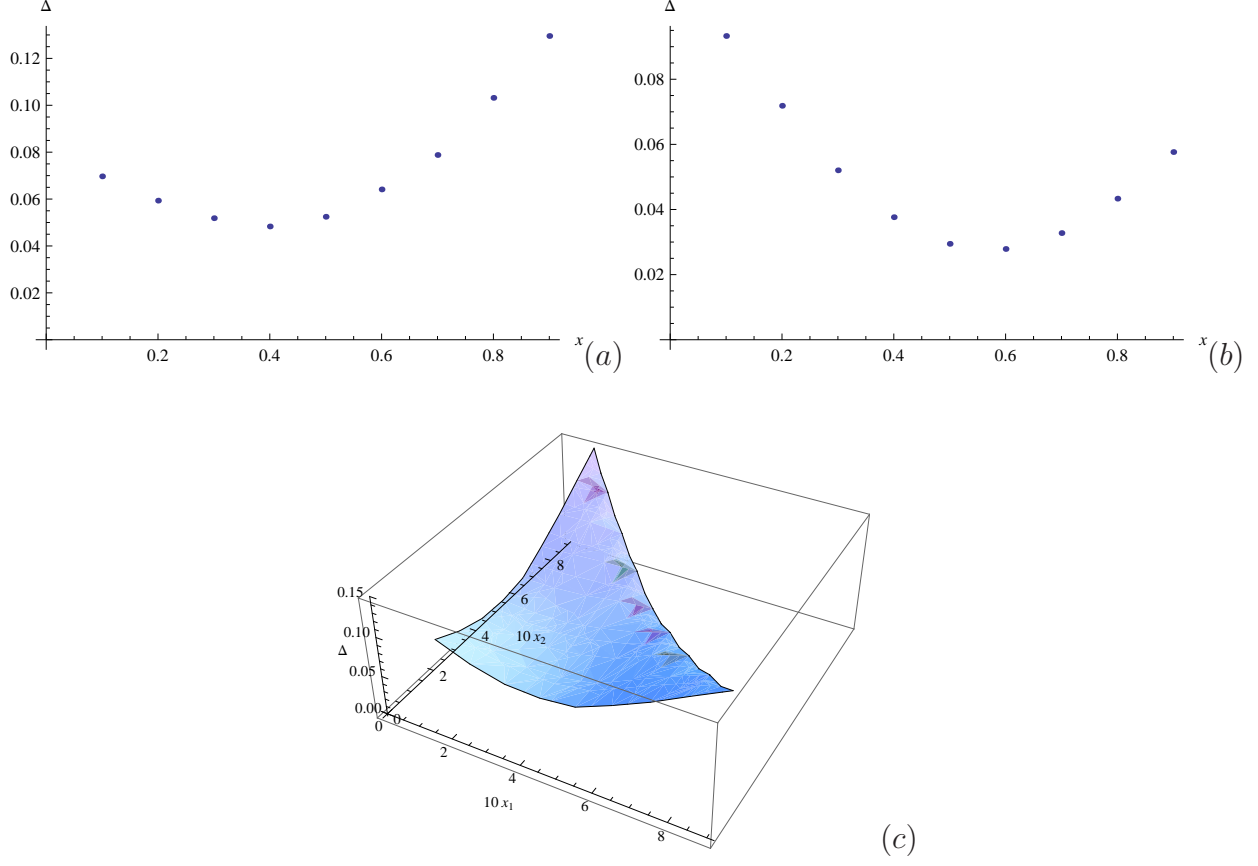


Figure 6: Plots expressing the function Δ in the case of the three RG flows described respectively in subsections 6.3.2, 6.3.3, and 6.3.4. The F-theorem requires that $\Delta > 0$, which is indeed verified. The results are based on the numerical solutions of section 2.

6.3.4. $\mathbf{A}_{11} \rightarrow \mathbf{A}_4 \otimes \mathbf{A}_4 \otimes \mathbf{A}_5$

Finally, we report a test of the F-theorem for a two-parameter family of RG flows defined as

$$(n = 10, \lambda) \rightarrow (n = 3, \lambda x_1) \otimes (n = 3, \lambda x_2) \otimes (n = 4, \lambda(1 - x_1 - x_2)) \quad (6.17)$$

with $0 < x_{1,2} < 1, 0 < 1 - x_1 - x_2 < 1$. In the case depicted in Fig. 6(c) $\lambda = 3$. The range of the conformal windows of the $\mathbf{A}_4, \mathbf{A}_{11}$ theories are

$$\mathbf{A}_4 : (0, 3) \quad , \quad \mathbf{A}_{11} : (2, 8) . \quad (6.18)$$

The F-theorem inequality (6.6) is more complicated in this case

$$\Delta(x_1, x_2) = f(3; 10) - x_1^2 f(3x_1; 3) - x_2^2 f(3x_2; 3) - (1 - x_1 - x_2)^2 f(3(1 - x_1 - x_2); 4) > 0 . \quad (6.19)$$

The results depicted in plot (c) of Fig. 6 verify this inequality.

One could proceed in this way to generate more tests of the F-theorem.

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Appendices

A. Multi-cut saddle point configurations

Our purpose in this appendix is to demonstrate that the saddle point equations (2.4) have general multi-cut solutions. We will discuss the presence of these solutions in a perturbative expansion of the equations in $\lambda \ll 1$ focusing, for illustration purposes, only on the leading term of the expansion. Our main concern here is not to find explicit solutions to the perturbative equations, but to demonstrate the allowed types of expansions. We have verified the existence of explicit solutions numerically.

A perturbative analysis of the saddle point equations

$$\mathcal{I}_i \equiv \frac{\mathbf{i}}{\lambda} t_i + \frac{1}{N} \sum_{j \neq i}^N \left[\coth(\pi t_{ij}) - \frac{(1-R) \sinh(2\pi t_{ij}) + t_{ij} \sin(2\pi R)}{\cosh(2\pi t_{ij}) - \cos(2\pi R)} \right] = 0, \quad i = 1, 2, \dots, N \quad (\text{A.1})$$

together with the R-extremization equation

$$\text{Re} \left[\sum_{i,j=1}^N (1-R + \mathbf{i} t_{ij}) \cot \pi(1-R + \mathbf{i} t_{ij}) \right] = 0 \quad (\text{A.2})$$

appeared in [9]. In this appendix we are interested in general values of R , hence we will not worry about the R-extremization equation.

In order to keep the discussion simple and compact we begin by listing a few characteristic expansions.

The standard one-cut solution crossing the origin

This expansion is based on the ansatz

$$t_i = \sqrt{\lambda} \left(t_i^{(0)} + \lambda t_i^{(1)} + \dots \right) , \quad i = 1, 2, \dots, N . \quad (\text{A.3})$$

Inserting this expansion into the saddle point equations (A.1) and expanding in powers of λ we get

$$\mathcal{I}_i = \frac{1}{\sqrt{\lambda}} \left(i t_i^{(0)} + \frac{1}{\pi N} \sum_{j \neq i}^N \frac{1}{t_i^{(0)} - t_j^{(0)}} \right) + \dots = 0 . \quad (\text{A.4})$$

The dots represent higher powers in the expansion. Ref. [9] explains how one can proceed to solve these equations systematically at any order. At leading order the equations (which are independent of R) reduce to those of the familiar Wigner model and the eigenvalues exhibit a semicircle distribution.

This expansion captures the one-cut solutions that reproduce correctly the perturbative field theory computation of the R-charge and are the key players in the numerical analysis of the main text.

Two-cut solutions crossing the imaginary axis at $\pm \frac{m i}{2}$

Consider an expansion based on the ansatz

$$\begin{aligned} t_i &= \frac{m i}{2} + \lambda t_i^{(0)} + \lambda^2 t_i^{(1)} , \quad i = 1, 2, \dots, \frac{N}{2} \in \mathcal{S}^+ , \\ t_i &= -\frac{m i}{2} + \lambda t_i^{(0)} + \lambda^2 t_i^{(1)} , \quad i = \frac{N}{2} + 1, \frac{N}{2} + 2, \dots, N \in \mathcal{S}^- . \end{aligned} \quad (\text{A.5})$$

Inserting this expansion into the saddle point equations we find for $m \in \mathbb{R}_+ - \mathbb{N}$

$$\mathcal{I}_{i \in \mathcal{S}^\pm} = \frac{1}{\lambda} \left(\mp \frac{m}{2} + \frac{1}{\pi N} \sum_{j \neq i \in \mathcal{S}^\pm} \frac{1}{t_i^{(0)} - t_j^{(0)}} \right) + \dots = 0 \quad (\text{A.6})$$

and for $m \in \mathbb{N}$

$$\mathcal{I}_{i \in \mathcal{S}^\pm} = \frac{1}{\lambda} \left(\mp \frac{m}{2} + \frac{1}{\pi N} \sum_{j \neq i}^N \frac{1}{t_i^{(0)} - t_j^{(0)}} \right) + \dots = 0 . \quad (\text{A.7})$$

One can show that only the $m \in \mathbb{N}$ case admits solutions. We have not attempted to find these equations analytically, but have verified numerically that they exist (see *e.g.* plot (b) of Fig. 1).

Two-cut solutions crossing the imaginary axis at $\pm \frac{(R+m)\mathbf{i}}{2}$

In this case we consider the ansatz

$$\begin{aligned} t_i &= \frac{(R+m)\mathbf{i}}{2} + \lambda t_i^{(0)} + \lambda^2 t_i^{(1)} , \quad i = 1, 2, \dots, \frac{N}{2} \in \mathcal{S}^+ , \\ t_i &= -\frac{(R+m)\mathbf{i}}{2} + \lambda t_i^{(0)} + \lambda^2 t_i^{(1)} , \quad i = \frac{N}{2} + 1, \frac{N}{2} + 2, \dots, N \in \mathcal{S}^- . \end{aligned} \quad (\text{A.8})$$

Inserting this expansion into the saddle point equations we obtain to leading order

$$\mathcal{I}_{i \in \mathcal{S}^\pm} = \frac{1}{\lambda} \left(\mp \frac{R+m}{2} + \frac{1}{\pi N} \sum_{j \neq i \in \mathcal{S}^\pm} \frac{1}{t_i^{(0)} - t_j^{(0)}} - \frac{m+1}{2\pi N} \sum_{j \in \mathcal{S}^\mp} \frac{1}{t_i^{(0)} - t_j^{(0)}} \right) + \dots = 0 . \quad (\text{A.9})$$

General multi-cut expansions

One can also entertain the possibility of more general multi-cut solutions based on the following ansatz

$$\begin{aligned} N_0 \text{ eigenvalues in } \mathcal{S}_0 &: t_i = \lambda^\alpha t_i^{(0)} + \dots , \\ N_m \text{ eigenvalues in } \mathcal{S}_m^\pm &: t_i = \pm \frac{m\mathbf{i}}{2} + \lambda t_i^{(0)} + \dots , \\ M_m \text{ eigenvalues in } \mathcal{S}_m^{\pm, R} &: t_i = \pm \frac{(R+m)\mathbf{i}}{2} + \lambda t_i^{(0)} + \dots , \end{aligned} \quad (\text{A.10})$$

so that

$$N_0 + \sum_{m=1}^{\infty} 2(N_m + M_m) = N . \quad (\text{A.11})$$

$\alpha = \frac{1}{2}, 1$ are exponents with special features but other values may also be allowed.

We have not attempted to solve the above equations analytically, but have verified numerically the existence of multi-cut solutions of the above more general type. In general, the actual solution of these equations will impose further constraints on the allowed values of the integers N_0, N_m, M_m .

The above list covers the full set of allowed expansions.

B. Spontaneous SUSY breaking implies $Z_{\mathcal{S}^3}^{(\text{loc})} = 0$: outline of an argument

In this appendix we outline a potential line of reasoning in favor of statement (b) in section 3. Our purpose is to show that in $\mathcal{N} = 2$ theories with spontaneous supersymmetry breaking the Q -deformed partition function $Z_{\mathcal{S}^3}^{(\text{loc})}$ vanishes. We describe on general grounds why this is a potentially correct generic property and highlight some of the key aspects that need to be clarified in order to reach a proof.

Let us begin with a few general comments. For a theory on a 3-manifold \mathcal{M} with an S^2 boundary there is a natural Hilbert space \mathcal{H} associated with the boundary S^2 [53]. The path integral of the theory over \mathcal{M} with boundary conditions $\Phi|_{S^2} = \chi$

$$Z(\chi) = \int_{\Phi|_{S^2} = \chi} e^{-S} \quad (\text{B.1})$$

defines a vector $|\chi\rangle$ in the Hilbert space \mathcal{H} . We are interested in the case where \mathcal{M} is a semisphere.

With the insertion of a generic operator \mathcal{O} at the pole of the semisphere the path integral with boundary conditions χ gives the amplitude $\langle \mathcal{O} | \chi \rangle$. This formalism is the basis of the operator-state correspondence (see, for example, [61] for a review). In a bra-ket language

$$Z(\chi) = \langle \Omega | \chi \rangle \quad (\text{B.2})$$

where $|\Omega\rangle$ is the vacuum state of the theory corresponding to the insertion of the identity operator.

The partition function on S^3 can be written in the following form

$$Z_{S^3} = \langle \Omega | \Omega \rangle = \int d\chi Z(\chi) Z^*(\chi) = \int d\chi \langle \Omega | \chi \rangle \langle \chi | \Omega \rangle . \quad (\text{B.3})$$

The second equality corresponds to gluing two 3-spheres at the S^2 equator and summing over the boundary conditions χ . We will be applying (B.3) to the Q -deformed path integral (3.1). In this language

$$\frac{dZ_{S^3}(t)}{dt} = -\langle \Omega | \int \{Q, V\} | \Omega \rangle . \quad (\text{B.4})$$

As a result, in a theory with spontaneously broken supersymmetry $Q|\Omega\rangle \neq 0$ and generically the derivative $\frac{dZ_{S^3}(t)}{dt}$ is non-zero. From now on we will be referring exclusively to the $t = \infty$ theory whose partition function is denoted as $Z_{S^3}^{(\text{loc})}$ in the main text.

Another useful piece of information that we need is the following. Localization is based on a supercharge Q . There is an additional supercharge Q^\dagger with anticommutator

$$\{Q, Q^\dagger\} = M + R \quad (\text{B.5})$$

which is also a symmetry of the theory. M , which is part of the $USp(2, 2)$ conformal group on S^3 , is a rotation on S^3 that can be viewed as a translation along the Hopf fiber of $S^1 \hookrightarrow S^3 \rightarrow S^2$. R is the R-symmetry operator.¹⁵

¹⁵For additional details we refer the reader to Ref. [2].

The ground states Ω have zero R-charge. Hence, using a standard argument based on the correlation function

$$\langle \Omega | M | \Omega \rangle = \langle \Omega | \{Q, Q^\dagger\} | \Omega \rangle = |Q|\Omega\rangle|^2 + |Q^\dagger|\Omega\rangle|^2 \quad (\text{B.6})$$

we deduce that a ground state is supersymmetric if and only if it has $M = 0$. Notice, however, that the second equation in (B.6) is strictly valid in Lorentzian signature. In Euclidean signature Q^\dagger is not the Hermitian conjugate of Q , but an independent supercharge. In what follows we will nevertheless assume that the above conclusion, *i.e.* that a ground state is supersymmetric if and only if it has $M = 0$, is also correct in the Euclidean theory. Clearly, this is a point that deserves further justification.

Furthermore, one can show, using an argument in [46], that the zero- M ground states are precisely the zero R-charge states $|\alpha\rangle$ that obey the equation

$$Q|\alpha\rangle = 0, \quad |\alpha\rangle \neq Q|\beta\rangle \quad (\text{B.7})$$

for any state $|\beta\rangle$. This concludes our short parenthesis of introductory comments.

By definition, $Z_{S^3}^{(\text{loc})}$ receives contributions from states $|\chi\rangle$ that solve the equation¹⁶

$$\{Q, V\}|\chi\rangle = 0. \quad (\text{B.8})$$

Applying Q to this equation and using $Q^2 = 0$ we find that the states $VQ|\chi\rangle$ are annihilated by Q , *i.e.* that

$$QVQ|\chi\rangle = 0. \quad (\text{B.9})$$

Therefore, according to the above general discussion, if the states $VQ|\chi\rangle$ are zero R-charge states that are not Q -exact we may infer that they have $M = 0$ and are therefore supersymmetric ground states. Then (b) follows naturally. When supersymmetry is spontaneously broken there are no supersymmetric ground states. Hence, there are no zero R-charge, zero- M ground states, including states of the form $VQ|\chi\rangle$, and therefore no states χ that obey equation (B.8) that can contribute to the path integral $Z_{S^3}^{(\text{loc})}$. As a result, $Z_{S^3}^{(\text{loc})}$ receives no contributions from the sum $\int d\chi$ in the rightmost expression in eq. (B.3) and vanishes identically as stated in (b).

The assumption that the states $VQ|\chi\rangle$ have zero R-charge is plausible for the following reason. The implementation of localization in general Yang-Mills theories with arbitrary

¹⁶More specifically, since localization forces the path integral on configurations Φ with the property $\{Q, V\}\Phi = 0$ the same property must be obeyed by continuity by the boundary conditions χ .

matter [1–3] and the eventual reduction of the path integral to a matrix integral over the real, zero R-charge scalar field σ in the $\mathcal{N} = 2$ vector multiplet, suggests that, at least in this general class of theories, χ is a state with zero R-charge. Since the R-charge of VQ is zero¹⁷ we conclude that $VQ|\chi\rangle$ is a zero R-charge state as well. In more general situations one has to examine precisely how localization works.

The other part of the argument requires showing that the state $VQ|\chi\rangle$ is not Q -exact, namely that there is no state $|\beta\rangle$ for which

$$VQ|\chi\rangle = Q|\beta\rangle . \quad (\text{B.10})$$

This property seems plausible, but we have not been able to find a rigorous proof.

It would be interesting to know if the specific choice of V plays any particular role in this argument. In general, $Z_{S^3}^{(\text{loc})}$ is expected to depend on the choice of V when a supersymmetric vacuum is absent, but the property $Z_{S^3}^{(\text{loc})} = 0$ may not. The latter would have to be true for (b) to hold in its current form, otherwise one would have to specify a special class of V functionals for which (b) is valid.

Moreover, part of our proposal is that any unknown trial R-charges in $Z_{S^3}^{(\text{loc})}$ should be fixed by $|Z_{S^3}^{(\text{loc})}|$ -minimization even in the absence of a supersymmetric vacuum. This prescription is motivated by the requirement to have a universally-prescribed quantity that reproduces the physical sphere partition function when supersymmetry is not broken. The implications and necessity of this prescription for the validity of (b) should be clarified.

C. Matching deformations in the \mathbf{A}_{n+1} duality

The duality (5.1) is expected to hold for arbitrary superpotential deformations of the type (6.1). Requiring the general matching of the vacuum structure on both sides of the duality gives useful information about the precise map between chiral ring operators. We proceed to derive this map by suitably adapting the four-dimensional analysis of [60].

We define the ‘electric’ theory as the $U(N)_k^{(n+1)}$ theory deformed by the general superpotential

$$W = \sum_{i=0}^n \frac{g_i}{n+1-i} \text{Tr } X^{n+1-i} . \quad (\text{C.1})$$

The dual ‘magnetic’ theory is a $U(nk - N)_{-k}^{(n+1)}$ theory deformed by a dual superpotential

¹⁷Indeed, the R-charge of the Q -deformation in (3.1) must be zero. The explicit formulae for V in [1–3] verify this property.

of the form

$$\overline{W} = \sum_{i=0}^n \frac{\bar{g}_i}{n+1-i} \text{Tr } \bar{X}^{n+1-i} + \alpha(\{g_i\}) . \quad (\text{C.2})$$

\bar{X} denotes the chiral superfield in the adjoint representation of the dual theory. We have allowed for a constant term in the superpotential denoted as $\alpha(\{g_i\})$. Our purpose is to determine \bar{g}_i and α as functions of the electric superpotential couplings g_i .

Electric theory

Consider, for example, the case where

$$W'(x) = \sum_{i=0}^n g_i x^{n-i} = g_0 \prod_{i=1}^n (x - a_i) \quad (\text{C.3})$$

with all a_i different. As described in the main text, in the deep IR the theory splits into a set of decoupled \mathbf{A}_2 theories with gauge groups $U(N_i)$ labelled by the sequence of integers

$$N_1 \leq N_2 \leq \dots \leq N_n , \quad \sum_{i=1}^n N_i = N . \quad (\text{C.4})$$

N_i eigenvalues of the matrix X reside in the i -th minimum of the potential $V = |W'(x)|^2$ labeled by a_i . Each of these \mathbf{A}_2 theories has a supersymmetric vacuum if and only if $N_i \leq k$.

Magnetic theory

Similarly, on the magnetic side

$$\overline{W}'(x) = \sum_{i=0}^n \bar{g}_i x^{n-i} = \bar{g}_0 \prod_{i=1}^n (x - \bar{a}_i) \quad (\text{C.5})$$

with all \bar{a}_i different. In the deep IR the theory splits into n copies of the \mathbf{A}_2 theory with gauge group $U(\bar{N}_i)$, such that

$$\bar{N}_1 \leq \bar{N}_2 \leq \dots \leq \bar{N}_n , \quad \sum_{i=1}^n \bar{N}_i = \bar{N} = nk - N . \quad (\text{C.6})$$

The duality must work individually for each of the n components of the IR theory, hence

$$\bar{N}_i = k - N_i , \quad i = 1, 2, \dots, n . \quad (\text{C.7})$$

For this property to be realized with any choice of partitions $\{N_i\}$ the electric and magnetic superpotentials must be closely related. In particular, whenever any number of critical

points a_i coincides, the same number of dual critical points \bar{a}_i must coincide as well. This requirement puts constraints on the dual couplings \bar{g}_i .

A potential solution to these constraints is to set

$$\bar{g}_i = c g_i, \quad i = 1, 2, \dots, n \quad (\text{C.8})$$

where c is some constant. We adopt this solution and, following [60], fix our conventions so that $c = -1$.

The duality map $g_i \rightarrow \bar{g}_i$ is closely related to the map between the chiral ring operators $\text{Tr } X^i$ and $\text{Tr } \bar{X}^j$. More information about this map can be inferred in the following way. Define the free energy $\mathbf{F}(g_i)$ of the theory in flat space as

$$e^{-\int d^3x d^2\theta \mathbf{F}(g_i) + c.c.} = \langle e^{-\int d^3x d^2\theta W(X, g_i) + c.c.} \rangle \quad (\text{C.9})$$

where g_i are background chiral superfields. Then

$$\frac{1}{n+1-i} \langle \text{Tr } X^{n+1-i} \rangle = \frac{\partial \mathbf{F}}{\partial g_i}. \quad (\text{C.10})$$

Similarly, on the magnetic side

$$\frac{1}{n+1-i} \langle \text{Tr } \bar{X}^{n+1-i} \rangle = \frac{\partial \bar{\mathbf{F}}}{\partial \bar{g}_i}. \quad (\text{C.11})$$

The duality implies

$$\bar{\mathbf{F}}(\bar{g}_i(g)) = \mathbf{F}(g_i). \quad (\text{C.12})$$

Hence,

$$\frac{1}{n+1-i} \text{Tr } X^{n+1-i} = \sum_{j=0}^n \frac{1}{n+1-j} \text{Tr } \bar{X}^{n+1-j} \frac{\partial \bar{g}_j}{\partial g_i} + \frac{\partial \alpha}{\partial g_i}. \quad (\text{C.13})$$

Taking the vev of both sides of this equation we find additional constraints on the map $g_i \rightarrow \bar{g}_i$. Most notably, the vevs of the left hand and right hand sides, which depend non-trivially on the particular vacuum (namely the partition $\{N_i\}$), must satisfy a relation independent of the particular vacuum. Implementing the ansatz (C.8) (with $c = -1$) we obtain the relation

$$\text{Tr } X^{n+1-i} = -\text{Tr } \bar{X}^{n+1-i} + (n+1-i) \frac{\partial \alpha}{\partial g_i}, \quad i = 1, 2, \dots, n. \quad (\text{C.14})$$

Taking the vev of both sides of this equation in a vacuum specified by the partition $\{N_i\}$ and the critical points $a_i = \bar{a}_i$ gives (taking also the duality eqs. (C.7) into account)

$$u_{n+1-i} = \frac{n+1-i}{k} \frac{\partial \alpha}{\partial g_i} \quad (\text{C.15})$$

where following [60] we have defined

$$u_j \equiv \sum_{i=1}^n (a_i)^j . \quad (\text{C.16})$$

Solving (C.15) one finds

$$\alpha = \frac{k}{n+1} \sum_{i=1}^n \left(i g_i \frac{u_{n+1-i}}{n+1-i} \right) . \quad (\text{C.17})$$

The following identities are useful in deriving this result

$$\frac{\partial}{\partial g_i} \left(\frac{u_{i+m}}{i+m} \right) = \text{independent of } i , \quad \sum_{i=1}^n \left(i g_i \frac{\partial u_j}{\partial g_i} \right) = j u_j . \quad (\text{C.18})$$

We conclude that the operator map takes the form

$$\text{Tr } X^{n+1-i} = -\text{Tr } \bar{X}^{n+1-i} + k u_{n+1-i} , \quad i = 0, 1, \dots, n \quad (\text{C.19})$$

and the superpotentials W, \bar{W} match with the identifications (C.8) and (C.17). In the special case where $g_i = 0$ ($i = 1, \dots, n$), $g_0 \neq 0$ the map (C.19) becomes simply

$$\text{Tr } X^{n+1-i} = -\text{Tr } \bar{X}^{n+1-i} , \quad i = 0, 1, \dots, n . \quad (\text{C.20})$$

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